Equilibrium States for Partially Hyperbolic Horseshoes

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Abstract

In this paper, we study ergodic features of invariant measures for the partially hyperbolic horseshoe at the boundary of uniformly hyperbolic diffeomorphisms constructed in [12]. Despite the fact that the non-wandering set is a horseshoe, it contains intervals. We prove that every recurrent point has non-zero Lyapunov exponents and all ergodic invariant measures are hyperbolic. As a consequence, we obtain the existence of equilibrium measures for any continuous potential. We also obtain an example of a family of C^∞ potentials with phase transition.

1 Introduction

Smale's horseshoes and geodesic flows in negative curved surfaces are, doubtless, landmarks in the study of dynamical systems theory. These examples are source for the general theory of *hyperbolic maps*, extensively studied from the structural and ergodic points of view. From the structural point of view, they are stable and open with respect to appropriate C^1 topology, exhibiting absolutely continuous invariant dynamical manifolds and they are sensitive with respect to small perturbations in initial conditions. From the ergodic point of view, there exist equilibrium measures for any continuous potential and every Hölder potentials admits an unique equilibrium state (on each transitive component). These measures are Gibbs states of the system, have full support, and their microscopic features are now very well understood.

To extend this theory beyond the hyperbolic setting is, nowadays, a very challenging task. In this direction, a very successful concept that extends the notion hyperbolicity, allowing some of its main consequences to be achieved, is the notion of *partial hyperbolicity*. This notion is a weak version of hyperbolicity that preserves many of its features. Many authors have successfully established results for partial hyperbolic systems: existence of absolutely continuous invariant manifolds ([7]), robust transitivity and generic properties ([13, 3, 4]), ergodic properties, as existence of SRB measures and stable ergodicity ([5, 1, 9]).

Concerning equilibrium states, some fruitful approaches by many authors have established existence or uniqueness beyond the hyperbolic setting, when the system under consideration has a specific structure. For instance, for interval maps, rational functions of the sphere, and Hénon-like maps we cite [8, 11, 19]; for countable Markov shifts and piecewise expanding maps, [10, 18, 20]; for horseshoes with tangencies at the boundary of hyperbolic systems, [15]; for higher dimensional local diffeomorphisms, [16, 2, 14], just to mention a few of the most recent works. Philosophically, a few restrictions can be imposed to the system to provide that all candidates for equilibrium measures have their exponents bounded away from zero. This gives to these non-uniformly hyperbolic maps the "flavor" of uniform hyperbolicity.

In this work we deal with a family of three dimensional partially hyperbolic horseshoes F at the boundary of uniformly hyperbolic diffeomorphisms. Each one of these horseshoes displays a heteroclinic cycles and its non-wandering set Λ contains intervals. They were constructed in [12] as time zero of bifurcations of families F_t of partially hyperbolic horseshoes. Here, we give a complete description of Lyapunov exponents in the central direction for ergodic measures, and prove that they are hyperbolic. As consequence of this, we get that *any* continuous potential admits equilibrium states.

Concerning uniqueness, we prove that the family $\phi_t = t\log|DF|_{E^c}|$ has a phase transition: there exists a $t_0 > 0$ such that ϕ_{t_0} admits at least two different equilibrium states. In view of the recent results of [17], it is likely that there is a unique equilibrium state for any t small enough. In fact, this seems to be true for any Hölder potential ϕ such that $\sup\phi - \inf\phi$ is smaller than some constant that depends only on the topological entropy and the expansion/contraction rates of F.

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1.1 Definition of the family of diffeomorphisms

We consider in $\mathbb{R}^3$ a family of horseshoe maps $F = F(\lambda_0, \lambda_1, \beta_0, \sigma, \beta_1)\colon R \to \mathbb{R}^3$, on the cube $R = I^3$, where I denote the interval $I = [0,1]$. Define the sub-cubes $R_0 = I \times I \times [0, 1/6]$, and $R_1 = I \times I \times [5/6, 1]$ of R. The restrictions F_i of F to R_i, $i = 0, 1$, are defined by:

- $F_0(x, y, z) = F_0(x, y, z) = (\lambda_0 x, f(y), \beta_0 z)$, with $0 < \lambda_0 < 1/3$, $\beta_0 > 6$ and f is the time one map of a vector field to be defined later;
- $F_1(x, y, z) = (3/4 - \lambda_1 x, \sigma(1-y), \beta_1(z - 5/6))$, with $0 < \lambda_1 < 1/3$, $0 < \sigma < 1/3$ and $3 < \beta_1 < 4$.

The map $f : I \to I$ is defined as the time one of the vector field

$$x' = -x(1-x).$$

This map is depicted in Figure 1.

Observe that $f(0) = 0$ and $f(1) = 1$, and , for every $y \neq 0$,

$$f^n(y) = \frac{1}{1 - \left(1 - \frac{1}{y}\right)e^{-n}}. \tag{1}$$

We also have

$$(f^n)'(y) = \frac{-\frac{e^{-n}}{y^2}}{\left(1 - \left(1 - \frac{1}{y}\right)e^{-n}\right)^2} = -\frac{e^{-n}}{y^2}\left(f^n(y)\right)^2. \tag{2}$$

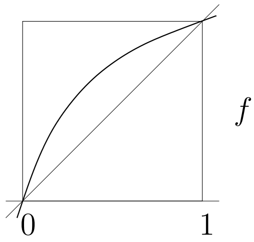

Figure 1: The central map f

Note that $f'(0) = e$ and $f'(1) = 1/e$. Since we have $f(0) = 0$ and $f(1) = 1$, the point $Q = (0, 0, 0)$ is a fixed saddle of index 1 of F, and the point $P = (0, 1, 0)$ is a fixed saddle of index 2 of F.

In [12], the authors proved that the diffeomorphism F is simultaneously at the boundary of the sets of uniformly hyperbolic systems and robustly non-hyperbolic systems. In fact, it is reached as the first bifurcation of a one-parameter family of C^∞ diffeomorphisms of the space $\mathbb{R}^3$, whose unfolding leads to robust non-hyperbolic behavior. Here we state some other properties of the diffeomorphism F, see [12] for proofs.

1. The diffeomorphism F has a heterodimensional cycle associated to the saddles P and Q.

2. The homoclinic class of Q is trivial and the homoclinic class of P is non-trivial and contains the saddle Q, thus $H(Q, F)$ is properly contained in $H(P, F)$.

3. There is a surjection

$$\Pi\colon H(P, F) \to \Sigma_{11}, \qquad \text{with} \quad \Pi \circ F = \sigma \circ \Pi,$$

   and infinitely many central curves C such that every C contains infinitely many points of the homoclinic class of $H(P, F)$ and $\Pi(x) = \Pi(y)$ for every pair of points $x, y \in C \cap H(P, F)$. These intervals consist of non-wandering points.

For the rest of the paper, Λ will denote the maximal invariant set in the cube. Namely

$$\Lambda = \bigcap_{n\in\mathbb{Z}} F^{-n}(R).$$

For $X = (x^s, x^c, x^u)$ in Λ, we denote by $W^u(X)$ and $W^s(X)$ the strong unstable and strong stable leaves of X. The central leaf $W^c(X)$, will denote the set of points on the form (x^s, y, x^u), with $y \in I$. Given an ergodic invariant measure μ we define the *central Lyapunov exponent* as:

$$\lambda^c_\mu = \int \log |DF|_{E^c}|\, d\mu.$$

Note that, since E^c is one dimensional and μ is ergodic,

$$\lambda^c_\mu = \lim_{n\to+\infty} \frac{1}{n} \log |DF^n(p)|_{E^c}|,$$

for μ almost every point $p \in \Lambda$.

2 Statement of the main results

Our first result is a description of central Lyapunov exponents for any ergodic invariant measure. We prove that the central Lyapunov exponents of these measures are negative, except for the measure δ_Q. Moreover, using this information, we are able to prove the existence of equilibrium measures associated to any continuous potential.

Theorem 2.1. *The following properties of F hold true:*

1. *For any recurrent point p different from Q:*
$$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^n(p)|_{E^c}| \leq 0.$$
Moreover, any ergodic invariant measure for F different from δ_Q has negative central Lyapunov exponent.

2. *If μ_n is a sequence of ergodic invariant measures such that λ^c_μ converges to zero, then μ_n converges to $\frac{\delta_Q+\delta_P}{2}$ in the weak* topology.*

Let $\phi : R \to \mathbb{R}$ be a continuous function. Let η be a F-invariant probability measure. The η-pressure of the potential ϕ (or equivalently the ϕ-pressure of the measure η) is defined by
$$h_\eta(f) + \int \phi \, d\eta.$$
We recall that η is called an *equilibrium state* for the potential ϕ if its ϕ-pressure maximizes the ϕ-pressures among all F-invariant probabilities. Our second result is:

Theorem 2.2. *Any continuous function $\phi : R \to \mathbb{R}$ admits an equilibrium state. Moreover, there exists a residual set of $C^0(\Lambda)$ such that the equilibrium measure is unique.*

If μ is an equilibrium state for some continuous potential ϕ, the ϕ-pressure of μ is also the *topological pressure* of ϕ. A natural question that arises from the previous theorem is if Hölder regularity of ϕ implies uniqueness of the equilibrium measure. A negative answer to this question for a particular potential is given in Theorem 2.3 below. Concerning this, we are able to prove that some restriction is necessary. We prove that $\phi_t = t \log |DF|_{E^c}|$ admits a phase transition:

Theorem 2.3. *Consider the one parameter family ϕ_t of C^∞ potentials defined for $X = (x, y, z) \in R$ by*
$$\phi_t(X) = t \log |DF(X)|_{E^c}| = \begin{cases} t \log f'(y), & \text{for } z \leq 1/6; \\ t \log \sigma, & \text{for } z \geq 5/6. \end{cases}$$
There exists a positive real number t_0 such that:

1. *For $t > t_0$, δ_Q is the unique equilibrium state.*

2. *For $t < t_0$, any equilibrium state for ϕ_t has negative central Lyapunov exponent. In particular, this measure is singular with δ_Q.*

3. *For $t = t_0$, δ_Q is an equilibrium state for ϕ_t, and there exists at least one other equilibrium state, singular with δ_Q.*

Remark 1. In fact, t_0 can be defined as the supremum, among all F-invariant measures different from δ_Q, of the expression $\left\{\frac{h_\mu(F)}{1-\lambda^c_\mu}\right\}$. Note that by Theorem 2.1, this number is well defined.

3 Central Lyapunov exponents

In this section we study some interesting features of F. We are able to prove that if $x \in \Lambda$ is recurrent and different from P and Q, then $W^c(X) \cap \Lambda = \{X\}$, despite the fact that Λ contains central intervals. We also prove that central Lyapunov exponents of any ergodic measure different from δ_Q is negative.

3.1 Central Lyapunov exponents for recurrent points

The main tool to prove the results in this section is the reduction of the dynamics to a one-dimensional system of iterated functions. Here we study these system, as well as some definitions and results in [12] that we need in this work.

Consider the maps $f_0, f_1 \colon I \to \mathbb{R}$ defined by

$$\begin{aligned} f_0(y) &= f(y), \\ f_1(y) &= \sigma\,(1-y). \end{aligned}$$

Given any $X = (x_0^s, x_0^c, x_0^u) \in \Lambda$ and $k \geq 0$, let $X_k = F^k(X) = (x_k^s, x_k^c, x_k^u)$. By the definition of F, the central coordinate x_k^c, of X_k is

$$x_k^c = f_{i_{k-1}} \circ f_{i_{k-2}} \circ \cdots \circ f_{i_0}(x_0^c),$$

where the numbers $i_0, \ldots, i_{k-1} \in \{0,1\}$ are determined by the coordinates $x_0^u, \ldots, x_{k-1}^u$. In fact, the map F admits a well defined projection along the central direction to I^2, and this projection is conjugated to the shift in Σ_{11}. Using this conjugacy, F can be thought as the skew product

$$\begin{aligned} \tilde{F} \colon \Sigma_{11} \times I &\to \Sigma_{11} \times I \\ (\theta, x) &\mapsto (\sigma\theta, f_\theta(x)), \end{aligned}$$

where $f_\theta = f_{\theta_0} \in \{f_0, f_1\}$.

In what follows, we consider the dynamics associated to the system of iterated functions (s.i.f.) generated by f_0 and f_1, that we denote by $\mathfrak{F}$.

Given a sequence $(i_n) \in \Sigma_{11}^+$, for each given $k \geq 0$ we consider the k-block $\varrho_k = \varrho_k(i_n) = (i_0, i_1, \ldots, i_k)$ associated to (i_n). For each k-block ϱ_k, we consider the map Φ_{ϱ_k} defined by

$$\Phi_{\varrho_k}(x) = f_{i_k} \circ f_{i_{k-1}} \circ \cdots \circ f_{i_0}(x).$$

The computation of the contraction in the central direction is based on an explicit computation of the derivative of the functions Φ_{ϱ_k}. First, consider a point $y \in (0,1]$. Then we have

$$|(f_1 \circ f_0^\alpha)'(y)| = \left(\frac{w}{y\,(1-y)}\right)\left(1 - \frac{w}{\sigma}\right), \quad \text{where } f_0^\alpha(y) = 1 - w/\sigma.$$

Note that $f_1 \circ f_0^\alpha(y) = w$. This implies that, if we chose a sequence $(i'_n) \in \Sigma_{11}^+$ such that (i'_n) is the concatenation of blocks of type $(0, \ldots, 0, 1)$, with the 1's occurring in the positions k_i, we have

$$\Phi'_{\varrho_{k_i}}(y) = \prod_{j=1}^{i} \frac{w_j\,(1 - w_j/\sigma)}{w_{j-1}\,(1 - w_{j-1})} \quad \text{where } w_0 = y \text{ and } w_j = \Phi_{\varrho_{k_j}}(y). \tag{3}$$

Observe that, if $w_j > 0$, the factor of the product in (3) corresponding to it is strictly smaller than 1. Moreover, it is a decreasing function of $w_j \in [0, \sigma]$.

Lemma 3.1. *Let $(i_n) \in \Sigma_{11}^+$ be a sequence with infinitely many 1's. Assume that $i_0 = 1$. Let $(n_j)_{j\geq 0}$ be the sequence of positions of the $j+1$'s symbols 1 for (i_n). Then, there exist a sequence of positive real numbers $(\delta_j)_{j\geq 0}$ and a positive constant C such that*

(i) for every y in $[0,1]$, $|\Phi'_{\varrho_{n_i}}(y)| \leq C \prod_{j=1}^{i-1} \frac{1-\delta_j/\sigma}{1-\delta_j}$,

(ii) C depends only on n_0,

(iii) each δ_j depends only on the n_i's, $i \leq j$.

Proof. Let ϱ' be the block of (i_n) starting at the first symbol and finishing at the second 1. Let $N = n_0$ be its size, and (i'_n) be the sequence obtained from (i_n) by removing ϱ'. Then, for $k > N$ and $y \in (0,1]$,

$$\Phi'_{\varrho_k}(y) = \Phi'_{\varrho'_{(k-N)}}(\Phi_{\varrho_N}(y)).\Phi'_{\varrho_N}(y). \tag{4}$$

Let $A = \max\{|\Phi'_{\varrho_N}(\xi)|, \xi \in I\}$. Note that A only depends on n_0.

Let $w_0 = \Phi_{\varrho_N}(y)$, and $w_j = \Phi_{\varrho'_{n_j-N}}(w_0)$. Observe that $\Phi_{\varrho_N}(I) \subset (0, \sigma]$; we set

$$\delta_0 = \min \Phi_{\varrho_N}(I) \text{ and } \delta_j = \min \Phi_{\varrho'_{n_j-N}}([\delta_0, \sigma]).$$

Then, (3) yields

$$|\Phi'_{\varrho'_{n_i-N}}(w_0)| = \frac{w_i\,(1-w_i/\sigma)}{w_0\,(1-w_0)} \prod_{j=1}^{i-1} \frac{1-w_j/\sigma}{1-w_j} \leq \frac{1}{3\delta_0(1-\delta_0)} \prod_{j=1}^{i-1} \frac{1-\delta_j/\sigma}{1-\delta_j}. \tag{5}$$

Therefore, (4) and (5) yield (i), with $C = \dfrac{A}{3\delta_0(1-\delta_0)}$. Note that C only depends on n_0. Moreover each δ_j only depends on the n_i's, with $i \leq j$. This finishes the proof of the lemma. ☐

Lemma 3.2. *Let $(i_n) \in \Sigma_{11}^+$ be a recurrent sequence for the shift such that $i_0 = 1$. Then there exist a real number a in $(0,1)$ and an increasing sequence of times $(m_j)_{j\geq 0}$ such that for every y in $[0,1]$,*

$$|\Phi'_{\varrho_{m_j}}(y)| \leq C.a^j,$$

where C is obtained from (i_n) as in Lemma 3.1.

Proof. Note that as the sequence (i_n) is recurrent, it has infinitely many symbols 1. We can thus apply Lemma 3.1. In particular, we use the notations of its proof.

Since each factor in the product in (iii)-Lemma 3.1 is strictly less than 1, it remains to show that there are infinitely many factors bounded from above by a number strictly smaller than 1. This is equivalent to show that there are infinitely many values of j such that δ_j is uniformly bounded away from zero.

The first block of ϱ' is composed by $n_1 - 1$ zeros and one 1. This implies that $\Phi_{\varrho'_{n_1-N}}[0,\sigma] \subset [f_1 \circ f_0^{n_1-1}(\sigma), \sigma]$, and so $\delta_1 > f_1 \circ f_0^{n_1-1}(\sigma)$. By the recurrence of the sequence (i'_n), this first block repeats infinitely many times. For each time j that it repeats, using the same argument, we conclude that $\delta_{j+1} \in [f_1 \circ f_0^{n_1-1}(\sigma), \sigma]$. This concludes the proof. ☐

Remark 2. A direct consequence of Lemma 3.2 is that any periodic point is hyperbolic, and if it is different from Q, it admits a negative Lyapunov exponent in the central direction.

Remark 3. The hypothesis "(i_n) recurrent" is not necessary, and it can be replaced by the weaker assumption: "One block of the form $(1, \underbrace{0\ldots,0}_{k}, 1)$, with a fixed k, appears infinitely many times in (i_n)".

3.2 Proof of Theorem 2.1

Let X be a recurrent point for F (for forward and backward iterations). Assume X is different from Q and P. Then X is forward-recurrent for F. Let us consider the one-sided sequence $\Pi(X)^+$, which is recurrent and admits infinitely many symbols 1. Hence, we can apply Lemma 3.2 to obtain

$$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^n(X)|_{E^c}| \leq 0.$$

This gives estimates for the forward iteration, but we can also get estimates for the backward iterations:

Proposition 3.3. *Let X be a recurrent point for F (for forward and backward iterations) different from Q and P. Then*

$$\cap_{n\in\mathbb{Z}} F^n(R) \cap W^c(X) = \{X\}.$$

Proof. Let $\Pi(X) = (i_n) \in \Sigma_{11}$. Without loss of generality, we can assume that $i_0 = 1$. Let ϱ_k denote any block $(i_0, \ldots, i_k)$ of the sequence (i_n). We denote by (i_{n^+}) the associated one-sided sequence. Again, we use vocabulary and notations from the proofs of Lemmas 3.1 and 3.2.

The infinite block $[(i_0, i_1, \ldots)]$ begins with the concatenation of the blocks ϱ_{n_0} and $\varrho'_{n_1-n_0}$. The constant C in Lemma 3.1 only depends on n_0. The sequence of m_j's in Lemma 3.2 is the sequence of appearances of the block $\varrho'_{n_1-n_0}$ ("shifted" to the end of the appearance).

By recurrence of (i_n), we know that the block $\varrho_{n_0}\varrho'_{n_1-n_0}$ appears infinitely many times in the sequence $(\ldots, i_{-2}, i_{-1}, i_0)$. We consider a decreasing sequence of integers $k_j \to -\infty$ such that $\sigma^{-k_j}((i_n))$ coincides with (i_n) at the positions $0, 1, \ldots, n_1$. We also ask that $k_j - k_{j+1} > n_1$. Lemma 3.2 implies that for every j and for every y in $[0,1]$,

$$|\Phi'_{[(i_{k_j}, \ldots, i_{-1})]}(y)| \leq C.a^j. \tag{6}$$

Let $L_j \subset I$ be the image of the interval I by the map $\Phi_{[(i_{n_j}, \ldots, i_{-1})]}$. Points in $\cap_{n\in\mathbb{Z}} F^n(R) \cap W^c(X)$ have their central coordinates belonging to the intersection of the sets L_j, $j > 0$. Now, (6) implies that the diameter of L_j converges to zero. We also have that each L_j is non-empty, compact and $L_{j+1} \subset L_j$. Thus, their intersection is a single point. This completes the proof of the proposition. $\square$

We define the *cylinder* associated with the block $\varrho = (i_0, \ldots, i_k)$ as follows:

$$[\varrho] = [i_0, \ldots, i_k] = \{x \in \Lambda; F^j(x) \in R_{i_j}, \text{ for } j = 0, \ldots, k\} = \bigcap_{j=0}^{k} F^{-j}(R_{i_j}) \cap \Lambda.$$

The last expression in the definition above tells us that these sets are always closed sets, since they are finite intersection of closed sets $F^{-j}(R_{i_j})$. We say that a point p has *positive frequency* for a set $A \subset \Lambda$ if

$$\gamma(p, A) = \liminf \frac{\#\{0 \leq j < n; f^j(p) \in A\}}{n} > 0.$$

Definition 3.4. *We say that a point p is of contractive type if*

$$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^n(p)_{|}E^c| < 0.$$

Next proposition is a tool to finish the proof of item *1* in Theorem 2.1.

Proposition 3.5. *Let $p \in \Lambda$ be a point with positive frequency $\gamma > 0$ at some cylinder associated with a l-block $\theta = (0, 0, \ldots, 0, 1)$. Then it is of contractive type and its central Lyapunov exponent is less than a constant $c(\gamma, l) < 0$ that depends only on γ and l.*

Proof. We simply use Lemma 3.2 and Remark 3. There exist a constant $C = C(p)$ and $a \in (0, 1)$, such that for every n satisfying $F^n(p) \in \theta$,

$$|DF^{n+l}(p)|_{E^c}| \leq C(p) a^{\#\{0 \leq j \leq n,\ F^j(p) \in A\}}. \tag{7}$$

Note that a depends only on the length of the cylinder θ, hence on l. Moreover, (7) yields

$$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^{n+l}(p)|_{E^c}| \leq \gamma(p, \theta) \log a < 0.$$

This finishes the proof of Proposition 3.5. ☐

Corollary 3.6. *Every ergodic and F-invariant probability μ which is not δ_Q has a negative Lyapunov exponent in the central direction.*

Proof. First, since every point in $[0, 0, \ldots] \setminus \{Q\}$ is attracted to P, the cylinder $[0, 0, \ldots]$ supports only two ergodic F-invariant measures, namely, δ_Q and δ_P. Thus, if μ is an ergodic measure different from δ_Q such that $\mu([0, 0, \ldots]) = 1$, μ must be δ_P. For this measure the Lyapunov exponent in the central direction equals -1.

Let us assume that $\mu([0, 0, \ldots]) < 1$. We claim that, if we define the k-block $\theta_k = [0, 0, \ldots, 0, 1]$, then there exist $\varepsilon > 0$ and $l \in \mathbb{N}$ such that $\mu([\theta_l]) > \varepsilon$. Indeed, just observe that

$$\Lambda \setminus \{[0, 0, \ldots]\} = \bigcup_{k=1}^{\infty} [\theta_k].$$

Thus, there exist some positive ε a l-block $\theta = [0, 0, \ldots, 0, 1]$ such that $\mu([\theta]) > \varepsilon$. By ergodicity, there exists a set of full μ-measure, $B_1 \subset \Lambda$, such that every $p \in B_1$ has frequency for θ equal to $\mu([\theta]) > \varepsilon > 0$. On the other hand, since μ is ergodic, there exists a set $B_2 \subset \Lambda$ with full μ-measure such that for every $p \in B_2$, the central Lyapunov exponent is well-defined and coincides with λ^c_μ. Taking any $p \in B_1 \cap B_2$ and observing Proposition 3.5, we have that $\lambda^c_\mu < c(\mu([\theta]), l) < 0$. ☐

Let us now prove item *2* in Theorem 2.1.

Proposition 3.7. *Let (μ_k) be a sequence of ergodic measures such that the sequence of central Lyapunov exponents $(\lambda^c_{\mu_k})$ converges to zero. Then, the sequence of measures (μ_k) converges to $\Delta = (1/2)\delta_Q + (1/2)\delta_P$.*

Proof. Given $\varepsilon > 0$ and $\theta_l = [0, \dots, 0, 1]$ an l-block, we define $E_{\varepsilon,l}$ by:

$$E_{\varepsilon,l} = \{\mu \text{ ergodic and } F\text{-invariant}; \mu(\theta_l) > \varepsilon\}.$$

From Corollary 3.6 and Proposition 3.5, there exists a constant $a = a(l) \in (0, 1)$, such that

$$\lambda^c_{\mu_k} = \mu_k([\theta_l]) \log a.$$

Therefore $\lim_{k\to+\infty} \mu_k([\theta_l]) = 0$. Since $[\theta_l]$ is open and closed in Λ, if μ is any accumulation point for the weak* topology, we get $\mu([\theta_l]) = 0$; this holds for every l, which means that $\mu([\theta_l]) = 0$ for any $l \in \mathbb{N}$. Hence, $\mu([0, 0, \dots]) = 1$, thus $\mu = \alpha\delta_Q + (1 - \alpha)\delta_P$, for some $\alpha \in I$.

Finally, we observe that $\log |DF|_{E^c}|$ is continuous; since μ is a weak* accumulation point for the sequence (μ_k), and $\lim_{k\to+\infty} \lambda^c_{\mu_k} = 0$, we get

$$0 = \int \log |DF|_{E^c}| \, d\mu = \alpha\lambda^c_{\delta_Q} + (1 - \alpha)\lambda^c_{\delta_P}.$$

Since $\lambda^c_{\delta_Q} = 1$ and $\lambda^c_{\delta_P} = -1$, we must have $\alpha = 1/2$. In particular, the sequence (μ_k) admits a unique accumulation point for the weak* topology. It thus converges to Δ and the proof is finished. $\square$

Remark 4. Using the structure provided by the heteroclinic cycle and the explicit expression of F, we can prove that there exists a sequence of periodic points p_n such that the Lyapunov exponents of the sequence of measures $\mu_n = (1/n)\sum_{i=0}^{n-1} \delta_{f^i(p_n)}$ converges to zero.

4 Proofs of theorems 2.2 and 2.3

4.1 Existence of equilibrium states

In this section we prove that the entropy function $\mu \to h_\mu(F)$ is upper-semicontinuous. As a consequence, we are able to prove the existence of equilibrium states for any continuous potential.

Observe that F is not a expansive map. It can be easily deduced observing that points in the central segment connecting Q and P have same α and ω limits, and F (respectively, F^{-1}) is a contraction when it is restricted to a neighborhood of P (respectively, Q). Nevertheless, we have the following:

Lemma 4.1. *Let μ be any F-invariant probability. Then every partition $\mathcal{P}$ with diameter smaller than $1/2$ is generating for μ.*

Proof. Let $\mathcal{P}$ be any partition with diameter smaller than 1/2. For any x in Λ we denote by $\mathcal{P}(x)$ the unique element of the partition which contains x. If n and m are two positive integers, we set

$$\mathcal{P}_{-m+1}^{n-1}(x) := \bigcap_{k=-m+1}^{n-1} F^{-k}(\mathcal{P}(F^k(x))),$$

and $\mathcal{P}_{-\infty}^{+\infty}(x)$ is the intersection of all $\mathcal{P}_{-m}^{n}(x)$. We have just to prove that for μ almost every point x, $\mathcal{P}_{-\infty}^{+\infty}(x) = \{x\}$.

Consider the set of recurrent points in Λ for F. This set has full μ-measure. Moreover, if x is recurrent, then its projection $\Pi(x)$ in Σ_{11} is also recurrent. If the bi-infinite sequence $\rho(x)$ contains at least one 1, Proposition 3.3 proves that $\mathcal{P}_{-\infty}^{+\infty}(x) \cap W^c(x) = \{x\}$. Hence, the uniform hyperbolicity in the two other directions yields $\mathcal{P}_{-\infty}^{+\infty}(x) = \{x\}$.

If the bi-infinite sequence $\Pi(x)$ does not contain any 1, then x must be in the segment $[Q,P]$. Therefore, $x = P$ or $x = Q$. Let us first assume that $x = Q$; then for any $y \in (Q,P]$, $\lim_{n\to+\infty} F^n(y) = P$. Hence,

$$\bigcap_{n\geq 0} F^{-n}(\mathcal{P}(Q)) \cap [Q,P] = \{Q\}.$$

Again, the uniform hyperbolicity in the two other directions yields $\mathcal{P}_{-\infty}^{+\infty}(Q) = \{Q\}$. If $x = P$, then for any $y \in [Q,P)$, $\lim_{n\to+\infty} F^{-n}(y) = Q$. The same argument yields $\mathcal{P}_{-\infty}^{+\infty}(P) = \{P\}$. $\square$

Following Proposition 2.19 in [6], we deduce that the metric entropy is a upper-semicontinuous function defined on a compact set. Thus, it attains its maximum. This imply the existence of equilibrium states for any continuous potential and uniqueness for any potential in a residual set of $C^0(M)$ is a standard matter, since $(\phi,\mu) \to h_\mu(f) + \int \phi\, d\mu$ is upper-semicontinuous on the set of invariant measures and is a convex function for $\phi \in C^0(M)$.

4.2 Phase transition: proof of Theorem 2.3

We denote by $\mathcal{P}(t)$ the topological pressure of $\phi_t = t\log|DF|_{E^c}|$. For convenience it is also referred as the topological t-pressure.

The function $t \mapsto \mathcal{P}(t)$ is convex, thus continuous on $\mathbb{R}$. Hence we can define $t_0 \leq +\infty$ as the supremum of the set

$$\mathcal{T} = \{\xi > 0,\ \forall\, t \in [0,\xi),\ \mathcal{P}(t) > t\}.$$

By continuity the set $\mathcal{T}$ is not empty because $\mathcal{P}(0) = h_{top}(F) > 0$.

Lemma 4.2. *For t in $[0,t_0)$, any equilibrium state μ_t for ϕ_t is singular with respect to δ_Q.*

Proof. Let us assume, by contradiction, that μ_t is an equilibrium state for ϕ_t with $\mu_t(\{Q\}) > 0$, for some $t \in [0,t_0)$. By the theorem of decomposition of measures, there exists a F-invariant measure ν, singular with respect to δ_Q such that $\mu_t = \mu_t(\{Q\})\delta_Q + (1-\mu_t(\{Q\}))\nu$. Since the metric entropy is affine, we have

$$\begin{aligned} t < \mathcal{P}(t) &= \mu_t(\{Q\})t + \big(1-\mu_t(\{Q\})\big)\left(h_\nu(F) + \int \phi_t\, d\nu\right) \\ &< \mu_t(\{Q\})\mathcal{P}(t) + (1-\mu_t(\{Q\}))\left(h_\nu(F) + \int \phi_t\, d\nu\right). \end{aligned}$$

In particular we get $\mathcal{P}(t) < h_\nu(F) + \int \phi_t\, d\nu$, which is absurd. □

Corollary 4.3. *Given t in $[0, t_0)$ and μ_t any equilibrium state for ϕ_t,*

$$\lambda^c_{\mu_t} = \int \log |DF|_{E^c}| d\mu_t < 0.$$

Proof. Let $\left(\nu_{t,\xi}\right)_{\xi\in\Lambda}$, be the ergodic decomposition of μ_t. Since $\mu_t(\{Q\}) = 0$, we have that for μ_t-almost every $\xi \in \Lambda$, $\nu_{t,\xi}(\{Q\}) = 0$. Corollary 3.6 says that for each of such ξ, we have $\displaystyle\int \log |DF|_{E^c}| d\nu_{t,\xi} < 0$. Therefore

$$\int \log |DF|_{E^c}| d\mu_t = \int_\Lambda (\int \log |DF|_{E^c}|\, d\nu_{t,\xi})\, d\mu_t(\xi) < 0.$$

□

Lemma 4.4. *The function $\mathcal{P}$ is decreasing on $[0, t_0)$.*

Proof. Let $t < t'$ be in $[0, t_0)$. Let us consider two equilibrium states for ϕ_t and $\phi_{t'}$, μ_t and $\mu_{t'}$. Then we have

$$\begin{aligned} \mathcal{P}(t') &= h_{\mu_{t'}}(F) + t'\lambda^c_{\mu_{t'}} \\ &= h_{\mu_{t'}}(F) + t\lambda^c_{\mu_{t'}} + (t'-t)\lambda^c_{\mu_{t'}} \\ &\le \mathcal{P}(t) + (t'-t)\lambda^c_{\mu_{t'}} \\ &< \mathcal{P}(t), \end{aligned}$$

where the last inequality yields from Corollary 4.3. □

Lemma 4.4 implies that $\mathcal{P}(t)$ is less than $h_{top}(F)$ on $[0, t_0)$. On the other hand, observe that $h_{\delta_Q}(F) + \int \phi_t\, d\delta_Q = t$, which means that $\mathcal{P}(t)$ is greater or equal to t. Therefore $t_0 \le h_{top}(F) < +\infty$ (see figure 2 for $t \le t_0$).

We can now finish the proof of Theorem 2.3. Note that existence of the real number t_0 and item *2* are already proved. By definition of t_0 and by continuity of $t \mapsto \mathcal{P}(t)$, we must have $\mathcal{P}(t_0) = t_0$, thus δ_Q is an equilibrium state for t_0. Moreover, any weak accumulation point for μ_t, as t *increases* to t_0, is an equilibrium state for t_0. Again, the continuity of $\log |DF|_{E^c}|$ yields that for such an accumulation point μ, the Lyapunov exponent λ^c_μ is non-positive, thus the measure is different from δ_Q.

Let us pick $t > t_0$. Let μ_t be any equilibrium state for t. We have

$$\begin{aligned} t \le \mathcal{P}(t) &= h_{\mu_t}(F) + t\lambda^c_{\mu_t} \\ &\le h_{\mu_t}(F) + t\lambda^c_{\mu_t} + (t-t_0)\lambda^c_{\mu_t} \\ t_0 + (t-t_0) &\le t_0 + (t-t_0)\lambda^c_{\mu_t}. \end{aligned}$$

This yields $\lambda^c_{\mu_t} \geq 1$. Again, considering the ergodic decomposition of μ_t, $(\nu_{t,\xi})$, we prove like in the proof of Corollary 4.3 that for almost every ξ, $\nu_{t,\xi} = \delta_Q$. In particular, this means that δ_Q is the unique equilibrium state for $t > t_0$ (see figure 2 for $t \geq t_0$). This complete the proof of Theorem 2.3.

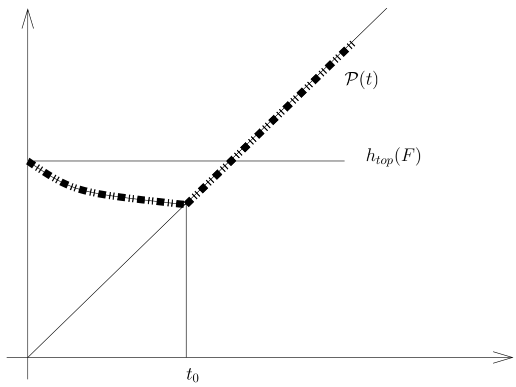

Figure 2: $t \mapsto \mathcal{P}(t)$

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